Morphisms of presheaves + sheaves (see Har II.1)

Let X be a topological space.

Def: If $\mathcal{F}$ and $\mathcal{G}$ are (pre)sheaves on X, a <u>morphism</u> $\varphi: \mathcal{F} \to \mathcal{G}$ consists of morphisms

$$\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open set, such that if $V \subseteq U$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

φ is an <u>isomorphism</u> if it has a two-sided inverse, i.e. some morphism ψ s.t. $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity on every open set U.

Claim: If $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves, there is an induced map of stalks

$$\varphi_p: \mathcal{F}_p \to \mathcal{G}_p$$

for all $p \in X$. (See HW1). Note that by construction, for any open $U \subset X$, and $p \in U$, this diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_p \arrow[r] & \mathcal{G}_p \end{tikzcd}$$

The additional requirements for a presheaf to be a sheaf allows us to check many properties locally:

Prop: A morphism of sheaves $\varphi: \mathcal{F} \to \mathcal{G}$ is an isomorphism $\iff$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for each $p \in X$.

Pf: $(\Rightarrow:)$ Follows from def of φ_p.

$(\Leftarrow:)$ Assume φ_p is an isomorphism for every $p \in X$.

It suffices to show $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for every open $U \subseteq X$.

For injectivity, take $s \in \mathcal{F}(U)$ s.t. $\varphi(U)(s) = 0$.
Then $\varphi(U)(s)_p = 0$ for each $p \in U$.

$$\begin{tikzcd} s \arrow[r, mapsto] & 0 \\ \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}_p \arrow[r] & \mathcal{G}_p \\ 0 & 0 \end{tikzcd}$$

Thus, $s_p = 0$ for each $p \in U$.
Thus, for each p, there is some open $V_p \subseteq U$ s.t. $s|_{V_p} = 0$. Since U is covered by the V_p, the sheaf condition tells us that $s = 0$ in $\mathcal{F}(U)$

For surjectivity, let $s \in \mathcal{G}(U)$. Then for each $p \in U$, $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective, so $\exists$ $t_p \in \mathcal{F}_p$ s.t.

$\varphi_p(t_p) = s_p$ (the germ of s at p)

Let t_p be represented by $t(P)$ on a neighborhood V_p of p. Then $\varphi(t(P))\big|_p = s_p$, so replacing V_p by a possibly smaller neighborhood, we can assume

$$\varphi(t(P)) = s\big|_{V_p}.$$

Thus, U is now covered by open sets V_p, and on each V_p, there is a section $t(P) \in \mathcal{F}(V_p)$. We need to check they are equal on intersections.

Let $Q \in U$, and $t(Q) \in \mathcal{F}(V_Q)$. Then

$$t(Q)\big|_{V_p \cap V_Q} \quad \text{and} \quad t(P)\big|_{V_p \cap V_Q}$$

both get sent, via φ, to $s\big|_{V_p \cap V_Q}$. By injectivity of φ, they must be equal (on $V_p \cap V_Q$).

Thus, the sheaf property tells us there is some $t \in \mathcal{F}(U)$ s.t. $t\big|_{V_p} = t(P)$ for each $P \in U$.

We just need to check $\varphi(t) = s$. For each V_p, we know that $\varphi(t)\big|_{V_p} = \varphi(t(P)) = s\big|_{V_p}$.

Thus, $(\varphi(t) - s)\big|_{V_p} = 0 \quad \forall\ V_p$, which forms an open cover of U.

Thus, $\varphi(t) - s = 0 \Rightarrow \varphi(t) = s$. $\square$

We've defined an isomorphism of sheaves in the obvious way, but injectivity and surjectivity are a bit more subtle. We first need the following:

Def: Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define the <u>presheaf kernel</u> of φ to be $U \mapsto \ker(\varphi(U))$, the <u>presheaf cokernel</u> to be $U \longmapsto \operatorname{coker}(\varphi(U))$, and the <u>presheaf image</u> to be $U \longmapsto \operatorname{im}(\varphi(U))$.

Remark: If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, the presheaf kernel is a sheaf, but the image and cokernel may not be.

However, there is a way to turn any presheaf (uniquely) into a sheaf as follows.

Prop/def: Let $\mathcal{F}$ be a presheaf on X. There is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ s.t. for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ s.t. $\varphi = \psi \circ \theta$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \mathcal{F}^+ \arrow[dl, dashed, "\exists!\, \psi"] \\ \mathcal{G} & \end{tikzcd}$$

$\mathcal{F}^+$ and θ are unique up to isomorphism, and $\mathcal{F}^+$ is called the <u>sheafification</u> of $\mathcal{F}$.

<u>Pf / construction:</u> Construct $\mathcal{F}^+$ as follows. For any open set U, define $\mathcal{F}^+(U)$ to be the set of functions $s : U \to \bigcup_{p \in U} \tilde{\mathcal{F}}_p$ s.t.

1.) for each $p \in U$, $s(P) \in \mathcal{F}_p$, and

2.) for each $P \in U$, there's a neighborhood V of P contained in U and $t \in \mathcal{F}(V)$ s.t. for all $Q \in V$, the germ t_Q of t at Q is $s(Q)$.

The restriction maps are just restrictions of functions.

The presheaf conditions follow immediately.

For the sheaf conditions, let $\{V_i\}$ be an open covering of U. Suppose $s \in \mathcal{F}^+(U)$ s.t. $s|_{V_i} = 0 \quad \forall i$.
Since restriction is given by restriction of functions, this means that $s(P) = 0 \quad \forall \; P \in U$, so $s = 0$.

Now assume $\forall i$ we have $s_i \in \mathcal{F}^+(V_i)$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for each i, j. Again, since each s_i is a function, there is a unique function $s \in \mathcal{F}^+(U)$ that has the requisite restrictions. It satisfies the desired properties by e.g. choosing the neighborhood of P to be contained within one of the V_i.

The morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ is given as follows: For $U \subseteq X$ open, and $s \in \mathcal{F}(U)$, $\theta(s) = (P \mapsto s_P)$.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism to a sheaf $\mathcal{G}$, we need to describe the morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$:

For $U \subseteq X$, $s \in \mathcal{F}^+(U)$, cover U by open sets V_i s.t. for each i there is a $t_i \in \mathcal{F}(V_i)$ s.t. the germ $t_Q = s(Q)$ for $Q \in V_i$.

Then since $\mathcal{G}$ is a sheaf, and $\varphi(t_i)$ agree on the overlaps, $\exists$ $t \in \mathcal{G}(U)$ s.t. $t|_{V_i} = \varphi(t_i)$. Set $\psi(s) = t$. It's straightforward to check that this is a morphism, and by construction it must be unique.

The uniqueness of $\mathcal{F}^+$ follows from this universal property. □

We'll also give another definition of $\mathcal{F}^+$ which is perhaps more intuitive:

(Alternate) def: Let $\mathcal{F}$ be a presheaf on X. We can define the sheafification of $\mathcal{F}$ as follows. For $U \subseteq X$ open,

$$\mathcal{F}^+(U) = \left\{ (s_P) \in \prod_{P \in U} \mathcal{F}_P \;\middle|\; (*) \right\}$$

where $(*)$ = for every $P \in U$ $\exists$ a neighborhood $V_P \subseteq U$ and $s \in \mathcal{F}(V_P)$ such that $\forall$ $Q \in V_P$, the germ of s at Q is s_Q.

Note: 1.) If $\mathcal{F}$ is already a sheaf then $\mathcal{F}^+ \cong \mathcal{F}$, and
2.) $\mathcal{F}^+_P = \mathcal{F}_P$ $\forall$ $P \in X$. (Check!!)

Subsheaves, kernels, images, and quotients

A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for each open $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps are induced by those of $\mathcal{F}$.

Note that since the presheaf kernel of a map $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is a sheaf, it is a subsheaf of $\mathcal{F}$. We'll call it simply the kernel of φ, or $\ker \varphi$, and say φ is injective if $\ker \varphi = 0$, i.e. if $\varphi(U)$ is injective for each $U \subseteq X$.

Define the image of φ, $\operatorname{im} \varphi$, to be the sheafification of the presheaf image of φ (call it $\mathcal{I}$).

Then by the universal property, $\exists$ a unique map $\operatorname{im}\varphi \to \mathcal{G}$ so that this commutes.

$$\begin{tikzcd} \lambda \arrow[r] \arrow[dr] & \mathcal{G} \\ & \operatorname{im}\varphi \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

In fact, it's injective, so $\operatorname{im}\varphi$ is a subsheaf of $\mathcal{G}$.

φ is <u>surjective</u> if $\operatorname{im}\varphi = \mathcal{G}$. Note that it may <u>not</u> be the case that $\varphi(U)$ is surjective for each U!!! i.e. injectivity can be checked on open sets, but not surjectivity.

If $\mathcal{F}'$ is a subsheaf of a sheaf $\mathcal{F}$, define the <u>quotient sheaf</u> $\mathcal{F}/\mathcal{F}'$ to be the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$.

Then the stalk of $\mathcal{F}/\mathcal{F}'$ at P is $\mathcal{F}_P/\mathcal{F}'_P$.

<u>Continuous maps and sheaves</u>

Let $f: X \to Y$ be a continuous map between topological spaces.

If $\mathcal{F}$ is a sheaf on X, the <u>direct image</u> of $\mathcal{F}$, denoted $f_*\mathcal{F}$ is defined

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for any open $U \subseteq Y$. ($f_* \mathcal{F}$ is commonly called the pushforward of $\mathcal{F}$.)

Ex: Let $p \in Y$ a fixed point, and define $f: X \to Y$ to be the constant map $f(x) = p \ \forall \ x \in X$.

If $\mathcal{F}$ is a sheaf on X, then for $U \subseteq Y$,

$$f_* \mathcal{F}(U) = \begin{cases} \mathcal{F}(X) & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$
(called a skyscraper sheaf)

More generally, for any open set disjoint from the image of a map the direct image of any sheaf will be 0 on that set.

On the other hand, if $\mathcal{G}$ is a sheaf on Y, how do we get a sheaf on X? If $U \subseteq X$ is open, $f(U)$ isn't necessarily, so we can't use an analogous definition. Instead, we take the limit of sections over open sets containing $f(U)$. i.e.:

Def: The inverse image sheaf $f^{-1}\mathcal{G}$ on X is the sheafification of the presheaf

$$U \longmapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

The inverse image is a lot harder to define, but it's way easier to see what the stalk is at a point:

$$(f^{-1}\mathcal{G})_x = \varinjlim_{U \ni x} f^{-1}\mathcal{G}(U) = \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}.$$

Def: If $Z \subseteq X$ is a subspace of X, and $i : Z \to X$ the inclusion map, and $\mathcal{F}$ a sheaf on X, then the <u>restriction of $\mathcal{F}$ to Z</u> is the sheaf $i^{-1}\mathcal{F}$, and is denoted $\mathcal{F}|_Z$.

Ex: If $x \in X$ is a point and $\mathcal{F}$ a sheaf on X. What is $\mathcal{F}|_{\{x\}}$?

$$\mathcal{F}|_{\{x\}} = \varinjlim_{U \supseteq \{x\}} \mathcal{F}(U) = \mathcal{F}_x.$$

That is, the restriction of a sheaf to a point is the stalk at that point.

Note: $f^{-1}\mathcal{G}$ is different than the pull-back $f^*\mathcal{G}$, which we'll define later.